Leech's Theorem and Factorization in Reproducing Kernel Hilbert spaces with Complete Nevanlinna-Pick Factor

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Abstract

The main purpose of the present thesis is to provide an alternative approach to a recent factorization theorem in [1], via an appropriate version of the famous theorem of Leech. The result regards Hilbert spaces with reproducing kernel factorized by a normalized complete Nevanlinna-Pick kernel. The novel aspect of this approach is a new proof of Leech's theorem in Hilbert spaces with normalized complete Nevanlinna-Pick reproducing kernel derived from an interpolation theorem. With this result in hand, we can extend Leech's theorem to the generalized context mentioned above, and then prove the factorization result we just mentioned.

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Contents

1 Introduction **4**

2 Preliminaries **6**
2.1 Scalar-Valued Hilbert Function Spaces 6
2.2 Examples of Hilbert function spaces 8
2.3 Vector-Valued Hilbert Function Spaces 11
2.4 Tensor product Hilbert spaces 13
2.5 Multipliers 15

3 Pick Interpolation and Leech's Theorem **18**
3.1 Normalized Complete Nevanlinna-Pick Kernels, Definition and Examples 18
3.2 Normalized Complete Nevanlinna-Pick Kernels, Basic Properties 19
3.3 Pick Interpolation on Arbitrary Subsets 22

4 An Application **27**
4.1 Kernels with normalized CNP factor, Definition and Examples . 27
4.2 Leech's theorem for kernels with normalized CNP factor 28
4.3 Proof of the factorization theorem 30

1 Introduction

In this thesis, reproducing kernel Hilbert spaces with the condition that the reproducing kernel has a normalized complete Nevanlinna-Pick factor are considered.

This condition is fulfilled for a large class of kernels, for example Hardy and weighted Bergman spaces on the unit ball $\mathbb{B}_d$, in $\mathbb{C}^d$, or the polydisc $\mathbb{D}^d$, and many other reproducing kernel Hilbert spaces. In fact, it turns out that every such space that has non-constant multipliers will have this property for some nontrivial normalized Nevanlinna-Pick kernel (see [1]).

The factorization result is motivated by a simple observation, which can already be seen at the simplest level of scalar kernels (see section 2 and section 2.5 below). For example, if the kernel k can be written as $k = sg$ with

$$s(z, w) = (1 - u(z)\overline{u(w)})^{-1}, \qquad g(z, w) = G(z)\overline{G(w)},$$

then by definition, G is a multiplier from the space with kernel s to the space with kernel k and as is well known, u is a multiplier of the space the space with kernel s into itself. This gives the factorization $k = \frac{\phi}{1-\psi}$, where ϕ, ψ are multipliers of the spaces mentioned above. The power of this result is that it extends this elementary observation for the kernel function, to the whole space of functions determined by this kernel. This is certainly a nontrivial task since in the generality considered here, we only know that the linear span of the kernels is dense in the space. Clearly, such a factorization result as above cannot follow only from this condition. The approach in [1] is constructive and based on the so-called Sarason function. It is also pointed out in [1] that an alternative approach can be based on the famous theorem of Leech [2], more precisely on an appropriate extension of that result to this context.

The purpose of this thesis is to accomplish this alternative approach. This is done by means of Leech's theorem for spaces with a normalized complete Nevanlinna-Pick kernel, which can be found in [3]. However, in this thesis we follow a different approach which is inspired by the ideas of Shimorin who proved a general (probably the most general) commutant lifting theorem for reproducing kernel Hilbert spaces [4], which we then combine with the powerful "realization theorem" of Agler and McCarthy [3, Theorem 8.30].

To be more specific the thesis contains a thorough and fairly rigorous presentation of the main mathematical objects involved in this circle of problems. In section 2 we introduce scalar and operator-valued Hilbert function spaces, their reproducing kernels, the multipliers on them, and tensor Hilbert spaces. In section 3 complete Nevanlinna-Pick kernels are presented, followed by a proof of Leech's theorem using Shimorin's approach. Kernels with a complete Nevanlinna-Pick factor are studied in section 4, which contains a generalization of Leech's theorem, leading to the proof of the factorization theorem.

The approach to Leech's theorem and the factorization theorem mentioned above consists of four steps. We begin with a theorem that we call "Pick interpolation on arbitrary subsets for Hilbert spaces with normalized complete Nevanlinna-Pick kernel". This result is a generalization of Pick's interpolation theorem obtained by answering the following question. Under which condition is it possible to find a multiplier of unit norm, taking prescribed values on a subset? A natural reformulation of the question is in terms of "lifting" adjoints of multipliers from specific invariant subspaces to the whole space, this brings us to Shimorin's work. With this interpolation theorem in hand, we obtain Leech's theorem for spaces with a normalized complete Nevanlinna-Pick kernels by Shimorin's approach. Afterwards, a far-reaching extension of Leech's theorem is proved for the case of an arbitrary reproducing kernel having a normalized complete Nevanlinna-Pick factor. Finally, with this general variant of Leech's theorem we present the proof of the factorization theorem mentioned at the beginning, which was the original purpose of this work.

2 Preliminaries

In this section the foundational theory of this thesis is laid out. We aim to give an introduction to vector-valued Hilbert function spaces, and their pointwise multipliers. First the scalar case is handled, following this will be the vector case. A detour to tensor products is necessary for the techniques used in this thesis, but will be very brief. Finally, we define pointwise multipliers in the general vector-valued case.

2.1 Scalar-Valued Hilbert Function Spaces

Throughout this thesis $\mathcal{X}$ is a fixed non-empty set. In this paragraph we shall be concerned with functions $k : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, and we begin by introducing the concept of what we call in this context, a kernel. It includes two properties defined below.

Definition 2.1. We say that the function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is *positive semi-definite* and write $k >> 0$ if for any finite subset $\{w_1, \dots, w_N\}$ of $\mathcal{X}$ and any complex numbers $c_1, \dots, c_N$

$$\sum_{i,j=1}^{N} \overline{c_i} c_j k(w_i, w_j) \geq 0.$$

Definition 2.2. The function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is said to be *self-adjoint* if

$$k(z, w) = \overline{k(w, z)} \qquad \forall z, w \in \mathcal{X}.$$

The central definition of this thesis is given below.

Definition 2.3. We say that the function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is a *kernel* on $\mathcal{X}$ if k is self-adjoint and $k >> 0$ and k is non-zero on the diagonal.

Kernels arise from certain Hilbert spaces of functions, which will be shown below.

Definition 2.4. A *(scalar-valued) Hilbert function space on $\mathcal{X}$* is a Hilbert space $\mathcal{H}$ of complex-valued functions on $\mathcal{X}$ where point evaluations are continuous non-zero linear functionals, that is, for each $w \in \mathcal{X}$ there exists $C_w \geq 0$ such that

$$|f(w)| \leq C_w \|f\|_{\mathcal{H}} \qquad \forall f \in \mathcal{H},$$

and there exists $f \in \mathcal{H}$ with $f(w) \neq 0$.

The main link between kernels and Hilbert function spaces is provided by the Riesz representation theorem [5, Theorem 3.4]. It implies that for each w in $\mathcal{X}$ there exists a unique function $k_w \in \mathcal{H}$ such that for all functions $f \in \mathcal{H}$

$$f(w) = \langle f, k_w \rangle_{\mathcal{H}}. \tag{1}$$

Definition 2.5. Let $\mathcal{H}$ be a Hilbert function space. The *reproducing function of $\mathcal{H}$* is the function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ that satisfies (1) for every $f \in \mathcal{H}$.

As usual, we shall alternate between the notations k_w and $k(\cdot, w)$. Furthermore the reproducing function is commonly called the reproducing kernel. We make this transition after we have proved that it is, in our meaning, a kernel.

Theorem 2.6. *Let $\mathcal{H}$ be a Hilbert function space on $\mathcal{X}$ and $k : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ be a function with the property that $k(\cdot, w) \in \mathcal{H}$ for all $w \in \mathcal{X}$. Then k is the reproducing function for $\mathcal{H}$ if and only if it satisfies*

$$k(z, w) = \langle k_w, k_z \rangle_{\mathcal{H}}, \qquad z, w \in \mathcal{X}, \tag{2}$$

the linear span of $\{k(\cdot, w) : \ w \in \mathcal{X}\}$ is dense in $\mathcal{X}$, and the function $(z, w) \mapsto k(z, w)$ is a kernel. In this case,

$$\|k_w\|^2 = k_w(w), \qquad w \in \mathcal{X}, \tag{3}$$

and if $\{e_n\}_{n\geq 1}$ is an orthonormal basis in $\mathcal{H}$, we have

$$k_w(z) = \sum_{n\geq 1} e_n(z)\overline{e_n(w)}, \qquad z, w \in \mathcal{X}.$$

Proof. Assume first that k is the reproducing function for $\mathcal{H}$. Then (2) is obvious. If $f \in \mathcal{H}$ is orthogonal to the linear span of $\{k(\cdot, w) : \ w \in \mathcal{X}\}$, then it satisfies $f(z) = \langle f, k_w \rangle_{\mathcal{H}} = 0$ for all $w \in \mathcal{X}$, i.e. $f = 0$. Consequently, the linear span of $\{k(w, \cdot) : \ w \in \mathcal{X}\}$ is dense in $\mathcal{H}$. To verify that k is a kernel, note that it is self-adjoint by (2). Moreover, let $\{w_1, \dots, w_N\}$ be a finite subset of $\mathcal{X}$ and $c_1, \dots, c_N$ some complex numbers. By (2)

$$\sum_{i,j=1}^{N} \overline{c_i} c_j k(w_i, w_j) = \|\sum_{j=1}^{N} c_j k_{w_j}\|^2_{\mathcal{H}} \geq 0,$$

hence $k_w(z) >> 0$. What remains to be shown is that k is non-zero on the diagonal. To this end, note first that (2) implies (3), hence if for some $w \in \mathcal{X}$, $k(w, w) = 0$, then $k_w = 0$ and if $f \in \mathcal{H}$,

$$f(w) = \langle f, k_w \rangle_{\mathcal{H}} = 0.$$

Which contradicts the assumption that the functions in $\mathcal{H}$ do not all share a common zero. The last equality in the statement follows from (2), since

$$\langle k_w, k_z \rangle_{\mathcal{H}} = \sum_{n\geq 1} \langle k_w, e_n \rangle_{\mathcal{H}} \langle e_n, k_z \rangle_{\mathcal{H}}.$$

Therefore, k is a kernel.

Conversely, if (2) holds and $f = \sum_j c_j k(w_j, \cdot)$ is a finite linear combination, then

$$f(w) = \sum c_j k(w_j, w) = \langle f, k_w \rangle_{\mathcal{H}}.$$

Since the set of these functions is dense in $\mathcal{H}$ and evaluation at w is continuous we obtain $f(w) = \langle f, k_w \rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$, and the proof is complete. □

Remark. Having proved that the reproducing function of a space $\mathcal{H}$ is a kernel, we call it the *reproducing kernel* or simply the kernel of $\mathcal{H}$.

According to this result, the structure of a Hilbert function space is completely encoded in its reproducing kernel. To decode this information from the kernel is a challenging task which is far from being understood, but leads to a rich theory combining Complex Analysis and Operator Theory.

The result below is a reformulation of the theorem we just proved.

Theorem 2.7. *There is a bijective correspondence between Hilbert function spaces on $\mathcal{X}$ and kernels on $\mathcal{X}$. Specifically, if k is a kernel on $\mathcal{X}$, then the Hilbert function space $\mathcal{H}$ on $\mathcal{X}$ obtained from k is the completion of the linear span of $\{k_w : w \in \mathcal{X}\}$ with respect to the norm induced by the unique inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ satisfying*

$$\langle k_w, k_z \rangle_{\mathcal{H}} = k(z, w), \qquad z, w \in \mathcal{X}.$$

This is the unique Hilbert function space on $\mathcal{X}$ with reproducing kernel k.

Due to this result Hilbert function spaces are commonly called *reproducing kernel Hilbert spaces*, and we introduce the following definition.

Definition 2.8. Let k be a kernel. $\mathcal{H}_k$ is defined to be the unique scalar Hilbert function space that has k as its reproducing kernel.

2.2 Examples of Hilbert function spaces

Example 2.9. *Sequence spaces.* ℓ^2, the space of sequences $(f(n))$ with

$$\|f\|^2 = \sum_n |f(n)|^2 < \infty$$

is a Hilbert function space on $\mathbb{N}$. Since the functions (sequences) f_k with $f_k(n) = \delta_{kn}$ form an orthonormal basis in ℓ^2, it follows by Theorem 2.6 that the reproducing kernel of this space is given by

$$k(j, l) = \sum_k f_k(j)\overline{f_k(l)} = f_j(l) = \delta_{jl}.$$

There is an obvious generalization of this example to weighted ℓ^2-spaces. More precisely, if $w = (w_n)$ is a sequence of positive numbers, let ℓ^2_w, be the space of sequences $(f(n))$ with

$$\|f\|_w^2 = \sum_n w_n |f(n)|^2 < \infty.$$

Clearly, this is a Hilbert function space on $\mathbb{N}$ as well. An orthonormal basis in this space is $\{f_k : f_k(n) = w_n^{-1/2}\delta_{kn}\}$. Therefore, by Theorem 2.6 the reproducing kernel for this space is given by

$$k(j, l) = \sum_k f_k(j)\overline{f_k(l)} = f_j(l) = w_l^{-1}\delta_{jl}.$$

Example 2.10. *Spaces of power series.* In many cases, sequence spaces can be identified with spaces of power series on the disc or even the unit ball in $\mathbb{C}^d$. Assume for simplicity that $w = (w_n)_{n \geq 0}$ is a sequence of positive numbers with

$$\lim_{n\to\infty} w_n^{\frac{1}{n}} = \lim_{n\to\infty} w_n^{-\frac{1}{n}} = 1.$$

Consider the space $\mathcal{H}_w$ of power series $f(z) = \sum_n f_n z^n$ in the unit disc $\mathbb{D}$ in the complex plane, such that $(f_n) \in \ell^2_w$ with

$$\|f\|_w = \|(f_n)\|_w.$$

Since the map $U : \ell^2_w \to \mathcal{H}_w$ with $U(f_n) = \sum_n f_n z^n$, is unitary, $\mathcal{H}$ is a Hilbert space. Moreover, for $f \in \mathcal{H}_w$ and $z \in \mathbb{D}$ we have

$$|f(z)| = \left|\sum_n f_n z^n\right| \le \left(\sum_n |z|^{2n} w_n^{-1}\right)^{\frac{1}{2}} \|f\|_w.$$

It follows from the assumption on w that $\mathcal{H}_w$ is a Hilbert function space. Its reproducing kernel is easily found with help of the unitary map U from above. Indeed, $g_n = U(w_n^{-1/2}\delta_{kn})$ form an orthonormal basis in $\mathcal{H}_w$, hence by Theorem 2.6

$$k(z,\zeta) = \sum_{n\ge 0} g_n(z)\overline{g_n(\zeta)} = \sum_{n\ge 0} w_n^{-1} z^n \overline{\zeta}^n$$

is the reproducing kernel of $\mathcal{H}_w$.

Some specific examples are:

1. The *Hardy space* obtained for $w_n = 1, n \ge 0$. Its reproducing kernel is $k(z,\zeta) = \frac{1}{1-z\overline{\zeta}}$, and is also called the Szegö kernel.

2. The *Bergman space* obtained for $w_n = \frac{1}{n+1}, n \ge 0$. The Bergman (reproducing) kernel is given by $k(z,\zeta) = \frac{1}{(1-z\overline{\zeta})^2}$.

3. The *Dirichlet space* obtained for $w_n = n+1, n \ge 0$, whose reproducing kernel is $\frac{1}{z\overline{\zeta}} \log \frac{1}{1-z\overline{\zeta}}$.

Example 2.11. *Power series on the unit ball.* A very similar construction can be carried out for power series on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$. Here we use positive weight sequences $(W_\alpha)_{\alpha\in\mathbb{N}_0^d}$ with the property that both series

$$\sum_\alpha W_\alpha r_1^{\alpha_1}\cdots r_d^{\alpha_d}, \qquad \sum_\alpha \frac{1}{W_\alpha} r_1^{\alpha_1}\cdots r_d^{\alpha_d},$$

converge whenever $r_1,\dots r_d > 0$ satisfy $\sum_{j=1}^d r_j^2 < 1$. As in the previous example we consider the space $\mathcal{H}_W$ consisting of power series

$$f(z) = \sum_\alpha f_\alpha z_1^{\alpha_1}\cdots z_d^{\alpha_d}$$

with the property that

$$\|f\|_W^2 = \sum_\alpha W_\alpha |f_\alpha|^2 < \infty.$$

With the same arguments as in the previous example it follows that $\mathcal{H}_W$ is a Hilbert function space on $\mathbb{B}_d$ and an orthonormal basis in $\mathcal{H}_W$ is given by the monomials $g_\alpha(z) = W_\alpha^{-1/2} z_1^{\alpha_1}\cdots z_d^{\alpha_d}$, $\alpha \in \mathbb{N}_0^d$. Therefore the reproducing kernel in $\mathcal{H}_W$ is given by

$$k(z,\zeta) = \sum_\alpha W_\alpha^{-1} z_1^{\alpha_1}\cdots z_d^{\alpha_d}\overline{\zeta_1}^{\alpha_1}\cdots\overline{\zeta_d}^{\alpha_d}.$$

This construction leads to $\mathbb{B}_d$-versions of the Hardy, Bergman, and Dirichlet space, but there are many other interesting examples. A reproducing kernel which received a lot of attention because of its applications in Operator Theory is the so called Drury-Arveson kernel

$$k(z,\zeta) = \frac{1}{1 - \langle z, \zeta \rangle_{\mathbb{C}^d}}.$$

The corresponding Hilbert function space is called the Drury-Arveson space. This kernel as well as the corresponding space can also be considered in the case when $d = \infty$.

Example 2.12. *Bergman spaces on general domains.* A direct calculation with the Parseval formula and polar coordinates on $\mathbb{D}$ shows that the norm in the Bergman space introduced in Example 2.10, which shall be denoted by L^2_a, is given by

$$\|f\|^2 = \sum_{n \geq 0} \frac{|f_n|^2}{n+1} = \frac{1}{\pi} \int_{\mathbb{D}} |f(x+iy)|^2 dxdy.$$

In other words, L^2_a consists of all holomorphic functions in $L^2(\mathbb{D})$ and is a closed subspace of it. The idea extends to other domains in one or several variables. Given an open connected subset Ω of $\mathbb{C}^d$, $L^2_a(\Omega)$ consists of those holomorphic functions f in Ω which belong to $L^2(\Omega)$. One can show that this is always a closed subspace of $L^2(\Omega)$, and that point evaluations are continuous on it. For large domains, for example if $\Omega = \mathbb{C}^d$, then the corresponding Bergman space is trivial ($L^2_a(\mathbb{C}^d) = \{0\}$). In the nontrivial situation the reproducing kernel of this space plays an important role in Analysis. S. Bergman considered simply connected domains in $\mathbb{C}$ and related the kernel to the Riemann map to the unit disc [6].

Example 2.13. *Dirichlet series.* A Dirichlet series is written formally as

$$f(s) = \sum_{n \geq 1} f_n n^{-s}.$$

If we assume for example, that $(f_n) \in \ell^2$ is a bounded sequence, then the series will converge uniformly on compacts in the half-plane $\{s : \operatorname{Re} s > \frac{1}{2}\}$. Let us consider (see [7]) the Hilbert space $\mathcal{H}^2$ consisting of Dirichlet series $f(s) = \sum_{n \geq 1} f_n n^{-s}$ with

$$\|f\|^2 = \sum_{n \geq 1} |f_n|^2 < \infty.$$

As in the previous examples it is easy to verify that $\mathcal{H}^2$ is a Hilbert function space on and that an orthonormal basis is given by $g_n(s) = n^{-s}$. Therefore, by Theorem 2.6 its reproducing kernel is given by

$$k(s,t) = \sum_{n \geq 1} n^{-s-\bar{t}}, \qquad \operatorname{Re} s, \operatorname{Re} s > \frac{1}{2}.$$

The right-hand side is the Riemann *zeta*-function evaluated at $s + \bar{t}$.

Example 2.14. *Sobolev spaces.* By the embedding theorems it follows that certain Sobolev spaces on $\mathbb{R}^d$ are actually Hilbert functions spaces. We want to discuss the simplest example in more detail. Consider the Hilbert space $\mathcal{H}$ consisting of absolutely continuous functions $f : [0,1] \to \mathbb{C}$ such that $f(0) = f(1) = 0$ and $f' \in L^2([0,1])$. The norm is defined by

$$\|f\|^2 = \int_0^1 |f'(t)|^2 dt, \qquad f \in \mathcal{H}.$$

Since for every $x \in [0,1]$

$$|f(x)| = \left| \int_0^x f'(t)dt \right| \leq x^{1/2} \|f\|,$$

we conclude that $\mathcal{H}$ is a Hilbert function space. In order to find the reproducing kernel we appeal to the Green function for the one-dimensional Laplacian. Intuitively it is clear by integration by parts that if k_x solves $-k_x'' = \delta_x$, where δ_x denotes the Dirac measure at x, then $\langle f, k_x \rangle_{\mathcal{H}} = f(x)$ for all $f \in \mathcal{H}$. The intuition is correct, since

$$k_x(y) = \begin{cases} (1-y)x \text{ if } x \leq y, \\ (1-x)y \text{ if } y \leq x \end{cases}$$

does the job.

2.3 Vector-Valued Hilbert Function Spaces

We now move to operator-valued kernels, which will naturally lead to Hilbert spaces of vector-valued functions.

Definition 2.15. Let $\mathcal{E}$ be a Hilbert space. We say that the function $K : \mathcal{X} \times \mathcal{X} \to \mathcal{B}(\mathcal{E})$ is *positive semi-definite* and write $K >> 0$ (or $K(z,w) >> 0$) if for any finite subset $\{w_1, \ldots, w_N\}$ of $\mathcal{X}$ and vectors $v_1, \ldots, v_N$ in $\mathcal{E}$

$$\sum_{i,j=1}^{N} \langle K(w_i, w_j) v_i, v_j \rangle_{\mathcal{E}} \geq 0.$$

Furthermore, let S also be a $\mathcal{B}(\mathcal{E})$-valued function on $\mathcal{X} \times \mathcal{X}$. We write $K(z,w) >> S(z,w)$ to mean that $K(z,w) - S(z,w) >> 0$ or equivalently

$$\sum_{i,j=1}^{N} \langle K(w_i, w_j) v_i, v_j \rangle_{\mathcal{E}} \geq \sum_{i,j=1}^{N} \langle S(w_i, w_j) v_i, v_j \rangle_{\mathcal{E}}.$$

Moreover, if $K >> 0$ and K is invertible on the diagonal, then we say that K is a *$\mathcal{B}(\mathcal{E})$-valued kernel on $\mathcal{X}$*.

Remark. Scalar-valued kernels on $\mathcal{X}$ can be seen as $\mathcal{B}(\mathbb{C})$-valued kernels on $\mathcal{X}$, as such, the theory developed in this subsection extends to scalar-valued kernels.

The vector-valued analogue of scalar-valued Hilbert function spaces is the following.

Definition 2.16. Let $\mathcal{E}$ be a Hilbert space. A *$\mathcal{E}$-valued Hilbert function space on $\mathcal{X}$* is a Hilbert space $\mathcal{H}$ of functions from $\mathcal{X}$ to $\mathcal{E}$ such that:

(i) For every $w \in \mathcal{X}$ evaluation at w is a continuous linear map from $\mathcal{H}$ to $\mathcal{E}$.

(ii) For every $w \in \mathcal{X}$ the linear span of $\{f(w) : f \in \mathcal{H}, w \in \mathcal{X}\}$ is dense in $\mathcal{E}$.

As in the scalar case, there is a bijective correspondence between $\mathcal{E}$-valued Hilbert function spaces on $\mathcal{X}$ and $\mathcal{B}(\mathcal{E})$-valued kernels on $\mathcal{X}$, which the following theorem ([3, Theorem 2.60]) shows. We shall omit the proof.

Theorem 2.17. *For a Hilbert space $\mathcal{E}$, there is a bijective correspondence between $\mathcal{E}$-valued Hilbert function spaces on $\mathcal{X}$ and $\mathcal{B}(\mathcal{E})$-valued kernels on $\mathcal{X}$. Specifically, if K is a $\mathcal{B}(\mathcal{E})$-valued kernel and $\mathcal{H}$ is the $\mathcal{B}(\mathcal{E})$-valued Hilbert function space obtained from K. Then this bijection is, such that, the linear span of $\{K_w v : w \in \mathcal{X}, v \in \mathcal{E}\}$ is dense in $\mathcal{H}$ and the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ satisfies*

$$\langle K_z u, K_w v\rangle_{\mathcal{H}} = \langle K(z,w)u, v\rangle_{\mathcal{E}}, \qquad z, w \in \mathcal{X}, u, v \in \mathcal{E}. \tag{4}$$

Many questions can be reduced to asking if a function is positive semi-definite. As such, we record the basics of the $>>$ order.

Proposition 2.18. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Let $\Phi : \mathcal{X} \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ and $K : \mathcal{X} \times \mathcal{X} \to \mathcal{B}(\mathcal{L}_1)$ with $K >> 0$ be given functions. Then $\Phi(z)K(z,w)\Phi(w)^*$ $>> 0$, in particular, letting $K = I_{\mathcal{E}}$ yields $\Phi(z)\Phi(w)^* >> 0$.*

Proof. Let $\{w_1, \ldots, w_N\}$ be a finite subset of $\mathcal{X}$. Let $u_1, \ldots, u_N$ be some vectors in $\mathcal{L}_1$ and $v_1, \ldots, v_N$ vectors in $\mathcal{L}_2$ such that $\Phi(w_i)^* u_i = v_i$. Then

$$\begin{aligned}\sum_{i,j=1}^{N} \langle \Phi(w_i)K(w_j, w_i)\Phi(w_j)^* u_j, u_i\rangle_{\mathcal{L}_1} &= \sum_{i,j=1}^{N} \langle K(w_j, w_i)v_j, v_i\rangle_{\mathcal{L}_2} \\ &\geq 0.\end{aligned}$$

□

Proposition 2.19. *Let $\mathcal{E}$ be a Hilbert space. Let K, L, S, and T be $\mathcal{B}(\mathcal{E})$-valued functions on $\mathcal{X} \times \mathcal{X}$. Then the following hold:*

$$K >> S \text{ and } L >> T \text{ imply } K + L >> S + T. \tag{5}$$

$$K >> L \text{ and } L >> S \text{ imply } K >> S. \tag{6}$$

$$K >> 0 \text{ and } S >> 0 \text{ imply } KS >> 0. \tag{7}$$

Furthermore

$$K >> S + T \text{ and } S >> 0, T >> 0 \text{ imply } K >> S \text{ and } K >> T. \tag{8}$$

We end this subsection with one particular class of operator-valued kernels that are of most interest. They arise from scalar-valued kernels and are the primary concern of this thesis.

Proposition 2.20. *Let k be a scalar-valued kernel on $\mathcal{X}$ and $\mathcal{E}$ a Hilbert space. Then $I_{\mathcal{E}}k$ is a $\mathcal{B}(\mathcal{E})$-valued kernel on $\mathcal{X}$.*

Proof. It is clear that $I_{\mathcal{E}}k$ is invertible on the diagonal. What remains to be shown is that $I_{\mathcal{E}}k >> 0$. To do this, let $\{w_1, \ldots, w_N\}$ be a finite subset of $\mathcal{X}$. Let $v_1, \ldots, v_N$ be some vectors in $\mathcal{E}$. Let $\{e_n\}_{n\geq 1}$ be the part of a basis for $\mathcal{E}$

such that $\langle v_i, e_j\rangle_{\mathcal{E}} \neq 0$ for all $1 \leq i \leq N$ and $j \in \mathbb{N}$ [5, Corollary 4.9]. Then by Parseval's theorem [5, Theorem 4.13. (e)]

$$\begin{aligned}\sum_{i,j=1}^{N} \langle k(w_i, w_j)v_j, v_i\rangle_{\mathcal{E}} &= \sum_{i,j=1}^{N}\sum_{n=1}^{\infty} k(w_i, w_j)\langle v_j, e_n\rangle_{\mathcal{E}}\langle e_n, v_i\rangle_{\mathcal{E}} \\ &= \sum_{n=1}^{\infty}\sum_{i,j=1}^{N} k(w_i, w_j)\langle v_j, e_n\rangle_{\mathcal{E}}\overline{\langle v_i, e_n\rangle}_{\mathcal{E}} \\ &\geq 0.\end{aligned}$$

□

We shall use the following notation for Hilbert function spaces with such kernels.

Definition 2.21. Let k be a scalar-valued kernel and $\mathcal{E}$ a Hilbert space. We denote the space which has $I_{\mathcal{E}}k$ as its kernel by $\mathcal{H}_k(\mathcal{E})$.

2.4 Tensor product Hilbert spaces

We briefly go through the basics of tensor-products Hilbert spaces, stating most propositions without proof. Specifically, we are interested in the relation of the tensor-product and the Hilbert function space $\mathcal{H}_k(\mathcal{E})$.

As is well known, the tensor product of Hilbert spaces is also a Hilbert space, which the following standard result shows [8].

Theorem 2.22. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. There exists a Hilbert space denoted by $\mathcal{L}_1 \otimes \mathcal{L}_2$, such that for each u in $\mathcal{L}_1$ and v in $\mathcal{L}_2$ there exists an element in $\mathcal{L}_1 \otimes \mathcal{L}_2$ written as $u \otimes v$, which is called an elementary tensor. This defines a map from $\mathcal{L}_1 \times \mathcal{L}_2$ to $\mathcal{L}_1 \otimes \mathcal{L}_2$ that has the following properties:*

$$\begin{aligned}(u_1 + u_2) \otimes v &= u_1 \otimes v + u_2 \otimes v, \\ u \otimes (v_1 + v_2) &= u \otimes v_1 + u \otimes v_2, \\ c(u \otimes v) &= cu \otimes v = u \otimes cv,\end{aligned}$$

where $u, u_1, u_2 \in \mathcal{L}_1$, $v, v_1, v_2 \in \mathcal{L}_2$, and $c \in \mathbb{C}$. Furthermore, the inner-product of $\mathcal{L}_1 \otimes \mathcal{L}_2$ satisfies

$$\langle u_1 \otimes v_1, u_2 \otimes v_2\rangle_{\mathcal{L}_1 \otimes \mathcal{L}_2} = \langle u_1, u_2\rangle_{\mathcal{L}_1}\langle v_1, v_2\rangle_{\mathcal{L}_2}.$$

Additionally, the linear span of elementary tensors are dense in $\mathcal{L}_1 \otimes \mathcal{L}_2$.

We have the following isomorphisms of tensor Hilbert spaces, which will be used without mention along with Proposition 2.30.

Proposition 2.23. *Let $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}$ be Hilbert spaces. Let $\cong$ mean 'is isomorphic to'. Then the following are true:*

$$\mathbb{C} \otimes \mathcal{L} \cong \mathcal{L}. \tag{9}$$

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L}_2 \otimes \mathcal{L}_1. \tag{10}$$

$$\mathcal{L}_1 \otimes (\mathcal{L}_2 \otimes \mathcal{L}) \cong (\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{L}. \tag{11}$$

$$\mathcal{L}_1 \otimes (\mathcal{L}_2 \oplus \mathcal{L}) \cong (\mathcal{L}_1 \otimes \mathcal{L}_2) \oplus (\mathcal{L}_1 \otimes \mathcal{L}). \tag{12}$$

The bounded linear operators on tensor Hilbert spaces relevant to us, are of the following type.

Proposition 2.24. *Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, and $\mathcal{L}_4$ be Hilbert spaces. Suppose we have operators $A \in \mathcal{B}(\mathcal{L}_1, \mathcal{L}_3)$ and $B \in \mathcal{B}(\mathcal{L}_2, \mathcal{L}_4)$. Then the map $A \otimes B : \mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{L}_3 \otimes \mathcal{L}_4$ defined on the linear span of elementary tensors by means of*

$$(A \otimes B)(u \otimes v) = Au \otimes Bv, \qquad u \in \mathcal{L}_1, v \in \mathcal{L}_2$$

extends to a bounded linear map from $\mathcal{L}_1 \otimes \mathcal{L}_2$ to $\mathcal{L}_3 \otimes \mathcal{L}_4$. Furthermore the adjoint satisfies

$$(A \otimes B)^* = A^* \otimes B^*.$$

Kernels can, as well as bounded linear operators on tensor Hilbert spaces be generated from the separate spaces.

Proposition 2.25. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Let K and S be functions from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{B}(\mathcal{L}_1)$ and $\mathcal{B}(\mathcal{L}_2)$, respectively, such that both $K >> 0$ and $S >> 0$. Then $K \otimes S$ is a $\mathcal{B}(\mathcal{L}_1 \otimes \mathcal{L}_2)$-valued function on $\mathcal{X} \times \mathcal{X}$ such that $K \otimes S >> 0$.*

Proof. Let $\{w_1, \ldots, w_N\}$ be a finite subset of $\mathcal{X}$. Let $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ be some vectors in $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then

$$\begin{aligned}
&\sum_{i,j=1}^{N} \langle (K(w_i, w_j) \otimes S(w_i, w_j))(u_i \otimes v_i), u_j \otimes v_j \rangle_{\mathcal{L}_1 \otimes \mathcal{L}_2} \\
&= \sum_{i,j=1}^{N} \langle K(w_i, w_j) u_i, u_j \rangle_{\mathcal{L}_1} \langle S(w_i, w_j) v_i, v_j \rangle_{\mathcal{L}_2} \\
&\geq 0.
\end{aligned}$$

□

The primary interest of this subsection is this following isomorphism.

Proposition 2.26. *Let k be a scalar-valued kernel and $\mathcal{E}$ a Hilbert space. Then $\mathcal{H}_k(\mathcal{E})$ is isomorphic to $\mathcal{H}_k \otimes \mathcal{E}$.*

Proof. Define the linear operator V on the linear span of $\{k_w \otimes u : w \in \mathcal{X}, u \in \mathcal{E}\}$ by means of

$$V(k_w \otimes u) = k_w u := f(\cdot)u, \qquad w \in \mathcal{X}, u \in \mathcal{E}.$$

By the properties of $\mathcal{B}(\mathcal{E})$-valued reproducing kernel Hilbert spaces and tensor Hilbert spaces

$$\begin{aligned}
\langle V(k_{w_1} \otimes u_1), V(k_{w_2} \otimes u_2) \rangle_{\mathcal{H}_k(\mathcal{E})} &= \langle k_{w_1} u_1, k_{w_2} u_2 \rangle_{\mathcal{H}_k(\mathcal{E})} \\
&= \langle k(w_2, w_1) u_1, u_2 \rangle_{\mathcal{E}} \\
&= \langle k_{w_1}, k_{w_2} \rangle_{\mathcal{H}_k} \langle u_1, u_2 \rangle_{\mathcal{E}} \\
&= \langle k_{w_1} \otimes u_1, k_{w_2} \otimes u_2 \rangle_{\mathcal{H}_k \otimes \mathcal{E}},
\end{aligned}$$

where $w_1, w_2 \in \mathcal{X}$ and $u_1, u_2 \in \mathcal{E}$. Hence V extends to a surjective isometry by Theorem 2.22 and Theorem 2.17. □

From this we now obtain a simple proof of the following proposition.

Proposition 2.27. *Let k be a scalar-valued kernel and $\mathcal{J}$ a closed subspace of a Hilbert space $\mathcal{L}$. Then $\mathcal{H}_k(\mathcal{J})$ is a closed subspace of $\mathcal{H}_k(\mathcal{L})$.*

Proof. By Proposition 2.26, it is enough to prove that $\mathcal{H}_k \otimes \mathcal{J}$ is a closed subspace of $\mathcal{H}_k \otimes \mathcal{L}$. From (12), the latter is isomorphic to $(\mathcal{H}_k \otimes \mathcal{J}) \oplus (\mathcal{H}_k \otimes \mathcal{J}^{\perp})$, this implies that it has $\mathcal{H}_k \otimes \mathcal{J}$ as a closed subspace. □

2.5 Multipliers

Definition 2.28. Let s and k be scalar-valued kernels. Let $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}$ be Hilbert spaces. Assume that the function $\Phi : \mathcal{X} \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ defines a bounded linear operator from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$ by

$$M_\Phi F = \Phi F := \Phi(\cdot)F(\cdot).$$

Then Φ is said to be a *multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$*, the set of these operator-valued functions is denoted by $\mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_k(\mathcal{L}_2))$. Moreover, we write $\|M_\Phi\| := \|M_\Phi\|_{\mathcal{B}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_k(\mathcal{L}_2))}$.

Finally, if $k = s$ and $\mathcal{L}_2 = \mathcal{L}_1 = \mathcal{L}$, then we say that Φ is a multiplier of $\mathcal{H}_k(\mathcal{L})$ and write $\Phi \in \mathbf{Mult}(\mathcal{H}_k(\mathcal{L}))$.

Remark. When $\mathcal{L}_2 = \mathcal{L}_1 = \mathbb{C}$, this coincides with the definition of multipliers from $\mathcal{H}_s$ to $\mathcal{H}_k$, in view of Proposition 2.26.

Proposition 2.29. *Let s and k be scalar-valued kernels. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces and $\mathcal{J}_2$ a closed subspace of $\mathcal{L}_2$. Suppose Φ is a multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{J}_2)$. Then Φ is a multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$, with the same norm.*

Proof. This follows directly from the definition of a multiplier and Proposition 2.27. □

Proposition 2.30. *Let s and k be scalar-valued kernels. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Suppose Φ is a $\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$-valued function on $\mathcal{X}$. Then Φ is a multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$ if and only if the operator T defined on the linear span of $\{k_w v : w \in \mathcal{X}, v \in \mathcal{L}_2\}$ by means of*

$$T k_w v = s_w \Phi(w)^* v, \qquad w \in \mathcal{X}, v \in \mathcal{L}_2$$

extends to a bounded linear operator from $\mathcal{H}_k(\mathcal{L}_2)$ to $\mathcal{H}_s(\mathcal{L}_1)$. In this case, $T = M_\Phi^*$.

Proof. Suppose Φ is a multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$. Then

$$\begin{aligned}
\langle fu, M_\Phi^* k_w v\rangle_{\mathcal{H}_s(\mathcal{L}_1)} &= \langle M_\Phi fu, k_w v\rangle_{\mathcal{H}_k(\mathcal{L}_2)} \\
&= \langle \Phi(w) f(w) u, v\rangle_{\mathcal{L}_2} \\
&= \langle f(w) u, \Phi(w)^* v\rangle_{\mathcal{L}_1} \\
&= \langle fu, s_w \Phi(w)^* v\rangle_{\mathcal{H}_s(\mathcal{L}_1)},
\end{aligned}$$

where $w \in \mathcal{X}$, $u \in \mathcal{L}_1$, $v \in \mathcal{L}_2$, and $f \in \mathcal{H}_s$. This shows that T is the restriction of M_Φ^* to the linear span of $\{k_w v : \ w \in \mathcal{X}, v \in \mathcal{L}_2\}$.

Conversely, suppose T extends to a bounded linear operator from $\mathcal{H}_k(\mathcal{L}_2)$ to $\mathcal{H}_s(\mathcal{L}_1)$. Then

$$\begin{aligned}\langle fu, Tk_w v\rangle_{\mathcal{H}_s(\mathcal{L}_1)} &= \langle fu, s_w \Phi(w)^* v\rangle_{\mathcal{H}_s(\mathcal{L}_1)}\\ &= \langle f(w)u, \Phi(w)^* v\rangle_{\mathcal{L}_1}\\ &= \langle \Phi(w) f(w) u, v\rangle_{\mathcal{L}_2},\end{aligned}$$

where $w \in \mathcal{X}$, $u \in \mathcal{L}_1$, $v \in \mathcal{L}_2$, and $f \in \mathcal{H}_s$. This shows that $T^*F = \Phi F$ on a dense subset of $\mathcal{H}_s(\mathcal{L}_1)$, therefore M_Φ is a bounded linear operator from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$. □

Proposition 2.31. *Let s and k be scalar-valued kernels. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Suppose Φ is a $\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$-valued function on $\mathcal{X}$. Then Φ is a multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$ with $\|M_\Phi\| \leq A$ if and only if*

$$A^2 I_{\mathcal{L}_2} k_w(z) - \Phi(z)\Phi(w)^* s_w(z) >> 0. \tag{13}$$

Proof. Define the linear operator T on the linear span of $\{k_w v : \ w \in \mathcal{X}, v \in \mathcal{L}_2\}$ by means of

$$Tk_w v = s_w \Phi(w)^* v, \qquad w \in \mathcal{X}, v \in \mathcal{L}_2.$$

Then (13) means that for a finite subset $\{w_1, \dots, w_N\}$ of $\mathcal{X}$ and some vectors $v_1, \dots, v_N$ in $\mathcal{L}_2$

$$\sum_{i,j=1}^{N} \langle A^2 k_{w_j}(w_i) v_j, v_i\rangle_{\mathcal{L}_2} \geq \sum_{i,j=1}^{N} \langle \Phi(w_i)\Phi(w_j)^* s_{w_j}(w_i) v_j, v_i\rangle_{\mathcal{L}_2},$$

this is equivalent to

$$A^2 \|\sum_{j=1}^{N} k_{w_j} v_j\|^2_{\mathcal{H}_k(\mathcal{L}_2)} \geq \|T \sum_{i=1}^{N} k_{w_j} v_j\|^2_{\mathcal{H}_s(\mathcal{L}_1)}.$$

Hence (13) is equivalent to $A\|f\|_{\mathcal{H}_k(\mathcal{L}_2)} \geq \|Tf\|_{\mathcal{H}_s(\mathcal{L}_1)}$, which in turn is equivalent to that T can be extended to a bounded linear operator on $\mathcal{H}_k(\mathcal{L}_2)$; this is equivalent to $\Phi \in \mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_k(\mathcal{L}_2))$ by Proposition 2.30. □

Lemma 2.32. *Let s and k be scalar-valued kernels. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, and $\mathcal{L}_4$ be Hilbert spaces. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic to $\mathcal{L}_3$ and $\mathcal{L}_4$, respectively, with isomorphisms $U_1 : \mathcal{L}_1 \to \mathcal{L}_3$ and $U_2 : \mathcal{L}_2 \to \mathcal{L}_4$.*

Then $\mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_k(\mathcal{L}_2)) \cong \mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_3), \mathcal{H}_k(\mathcal{L}_4))$. In the sense that for each Φ in $\mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_k(\mathcal{L}_2))$, then $U_1 \Phi U_2^* \in \mathbf{Mult}(\mathcal{H}_s(\mathcal{F}), \mathcal{H}_k(\mathcal{E}))$, moreover $\|M_{U_1\Phi U_2^*}\| = \|M_\Phi\|$.

Proof. Suppose that $\Phi \in \mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_k(\mathcal{L}_2))$ with $\|M_\Phi\| \leq A$. By Proposition 2.31 and since U_2 is an isomorphism

$$A^2 I_{\mathcal{L}_2} k_w(z) - \Phi(z) U_2 U_2^* \Phi(w)^* s_w(z) = A^2 I_{\mathcal{L}_2} k_w(z) - \Phi(z)\Phi(w)^* s_w(z) >> 0.$$

By Proposition 2.18, the above statement is equivalent to

$$\begin{aligned}&U_1 A^2 I_{\mathcal{L}_2} k_w(z) U_1^* - U_1 \Phi(z) U_2 U_2^* \Phi(w)^* s_w(z) U_1^*\\ &= A^2 I_{\mathcal{E}} k_w(z) - U_1 \Phi(z) U_2 U_2^* \Phi(w) U_1^* s_w(z) >> 0\end{aligned}$$

since U_1 is an isomorphism. Hence (13) is satisfied for $U_1 \Phi U_2^*$ and $\|M_{U_1\Phi U_2^*}\| \leq A$. Therefore $\|M_{U_1\Phi U_2^*}\| \leq \|M_\Phi\|$ and the reverse inequality is obtained from repeating the argument above for the operator $U_1^* U_1 \Phi U_2^* U_2 = \Phi$. □

Remark. It is clear that compositions of multipliers are multipliers. Specifically, let s, t, and k be scalar-valued kernels. Let $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ be Hilbert spaces. Suppose we have two functions $\Phi_1 \in \mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_t(\mathcal{L}_2))$ and $\Phi_2 \in \mathbf{Mult}(\mathcal{H}_t(\mathcal{L}_2), \mathcal{H}_k(\mathcal{L}_3)$. Then the composition $\Phi_2\Phi_1$ is a multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$ with norm satisfying $\|M_{\Phi_2\Phi_1}\| \leq \|M_{\Phi_2}\|\|M_{\Phi_1}\|$. This fact will be used without mention.

Lemma 2.33. *Let s and k be scalar-valued kernels. Let $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$ be Hilbert spaces. Suppose Ψ is a contractive multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2 \oplus \mathcal{L}_3))$. Then Ψ has the decomposition*

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

where Ψ_1 and Ψ_2 are contractive multipliers from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$ and $\mathcal{H}_k(\mathcal{L}_3)$, respectively.

Proof. For $z \in \mathcal{X}$, the linear operator $\Psi(z) \in \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2 \oplus \mathcal{L}_3)$ has decomposition

$$\Psi(z) = \begin{pmatrix} \Psi_1(z) \\ \Psi_2(z) \end{pmatrix}$$

where $\Psi_1(z)$ and $\Psi_2(z)$ are bounded linear operators from $\mathcal{L}_1$ to $\mathcal{L}_2$ and $\mathcal{L}_3$, respectively. Plugging in this decomposition into (13) yields

$$I_{\mathcal{L}_2} k_w(z) - [\Psi(z)\Psi(w)^* + \Theta(z)\Theta(w)^*] s_w(z) >> 0.$$

This implies that Ψ_1 and Ψ_2 are contractive multipliers from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_k(\mathcal{L}_2)$ and $\mathcal{H}_k(\mathcal{L}_3)$, respectively, by Proposition 2.18 and (8). $\square$

3 Pick Interpolation and Leech's Theorem

In this section we present the extension of an important theorem proved by Leech for vector-valued Hardy spaces on the unit disc, to the context of arbitrary Hilbert spaces with a normalized complete Nevanlinna-Pick reproducing kernel. The definition and basic properties of such kernels are discussed below.

3.1 Normalized Complete Nevanlinna-Pick Kernels, Definition and Examples

The definition of complete Nevanlinna-Pick kernels is rather lengthy and involved, since it goes back to the Pick interpolation problem and matrix versions of it. Therefore, we shall rather appeal to a characterization of such kernels that are normalized, obtained by Agler and McCarthy in [3, Theorem 7.31], and use their theorem to define the objects in question.

Definition 3.1. A scalar-valued kernel s on the set $\mathcal{X}$ is called a Complete Nevanlinna-Pick (CNP) kernel normalized at $z_0 \in \mathcal{X}$ if there exists an auxiliary Hilbert space $\mathcal{K}$ and a function $b : \mathcal{X} \to \mathcal{B}(\mathcal{K}, \mathbb{C})$, such that

$$s(z,w) = \frac{1}{1 - \langle b(z), b(w) \rangle_{\mathcal{K}}} \tag{14}$$

with $b(z_0) = 0$. Likewise we call s a normalized CNP kernel if we do not need to specify the point of normalization. Moreover, s is called a (not necessarily normalized) CNP kernel if it is non-zero and if the function $\tilde{s}$ on $\mathcal{X} \times \mathcal{X}$ defined by

$$\tilde{s}(z,w) = \frac{s(z_0, z_0)s(z,w)}{s(z,z_0)s(z_0,w)}$$

is a CNP kernel normalized at z_0.

Example 3.2. *Kernels of spaces of power series.* In example 2.10 we considered Hilbert function spaces on $\mathbb{D}$, whose reproducing kernel has the form

$$k(z,w) = \sum_{n \geq 0} w_n^{-1} z^n \overline{w}^n.$$

where $(w_n)_{n\geq 0}$ is a sequence of positive numbers with

$$\lim_{n\to\infty} w_n^{\frac{1}{n}} = \lim_{n\to\infty} w_n^{-\frac{1}{n}} = 1.$$

Clearly, these kernels are normalized at 0 if and only if $w_0 = 1$. Agler and McCarthy [3, Theorem 7.33] characterized the sequences $(w_n)_{n\geq 0}$, with the property that the resulting kernel is CNP. They prove that k is CNP if and only if

$$\frac{1}{k(z,w)} = 1 + \sum_{n\geq 1} c_n z^n \overline{w}^n,$$

with

$$c_n \leq 0 \qquad \forall n \geq 1.$$

Example 3.3. *The Szegö kernel.* Recall from example 2.10 that the reproducing kernel for the Hardy space on the unit disc, usually denoted by H^2, is the Szegö kernel, obtained for $w_n = 1, n \geq 0$, that is,

$$k(z,w) = \sum_{n\geq 0} z^n \overline{w}^n = \frac{1}{1 - z\overline{w}}.$$

Hence, the Szegö kernel is a CNP kernel normalized at 0 by definition.

Example 3.4. *The Bergman kernel.* Similarly the Bergman space L_a^2 is obtained for $w_n = \frac{1}{n+1}, n \geq 0$, as in example 2.10. In this case its reproducing kernel, or the Bergman kernel, is given by

$$k(z,w) = \sum_{n\geq 0} \frac{z^n \overline{w}^n}{n+1} = \frac{1}{(1 - z\overline{w})^2}.$$

Note that it is the square of the Szegö kernel, and its reciprocal satisfies

$$\frac{1}{k(z,w)} = (1 - z\overline{w})^2 = 1 - 2z\overline{w} + z^2\overline{w}^2.$$

Thus, while it is normalized, it is *not* a CNP kernel since the coefficient of the squares is positive.

Example 3.5. *The Drury-Arveson kernel.* Recall from example 2.11 that the Drury-Arveson kernel is given by

$$k(z,w) = \frac{1}{1 - \langle z, w\rangle_{\mathbb{C}^d}}.$$

Which means that it is a CNP kernel normalized at 0, by definition.

Example 3.6. *The Riemann zeta-function.* The reproducing kernel of the space $\mathcal{H}^2$ introduced in example 2.13 was shown to be given by

$$k(s,t) = \sum_{n\geq 1} n^{-s-\bar{t}}, \qquad \text{Re } s, \text{Re } t > \frac{1}{2},$$

where the right-hand-side is the Riemann zeta-function evaluated at $s + \bar{t}$. Furthermore it is well known that the reciprocal of the zeta-function satisfies

$$\frac{1}{k(s,t)} = \sum_{n\geq 1} \mu(n) n^{-s-\bar{t}}, \qquad \text{Re } s, \text{Re } t > \frac{1}{2},$$

where μ is the Möbius function. There is an analogous categorization of CNP kernels for spaces of Dirichlet series to the one discussed above, proved by J.E. McCarthy and M. Shalit [9, Theorem 26] which implies that k is not a CNP kernel since $\mu(n)$ is positive for some $n \geq 2$.

3.2 Normalized Complete Nevanlinna-Pick Kernels, Basic Properties

We record some basic results regarding normalized CNP kernels, in fact we change (14) to a more suitable form. But first, an introduction to the conjugate in Hilbert spaces is needed.

Definition 3.7. Fix a basis $\{e_i\}_{i\in\mathcal{I}}$ of the Hilbert space $\mathcal{E}$. Let v be an element of $\mathcal{E}$ and let $\{e_n\}_{n\geq 1}$ be the at-most countable collection of the basis vectors that have non-zero inner product with v [5, Corollary 4.9]. Then *the conjugate of* v, $\overline{v}$ is given by

$$\overline{v} = \sum_{n=1}^{\infty} \overline{\langle v, e_n\rangle_{\mathcal{E}}} e_n.$$

Using Parseval's theorem, one can prove the equality

$$\langle \overline{v}, \overline{u}\rangle = \overline{\langle \overline{u}, \overline{v}\rangle_{\mathcal{E}}} = \langle u, v\rangle_{\mathcal{E}}, \qquad u, v \in \mathcal{E}. \tag{15}$$

Remark. Whenever the conjugate is used, we presuppose that there is an underlying fixed basis.

Proposition 3.8. *Let s be a CNP kernel normalized at z_0 given in the form* (14)*. Define the function $u : \mathcal{X} \to \mathcal{B}(\mathcal{K}, \mathbb{C})$ by*

$$u(z)v = \langle v, \overline{b(z)}\rangle, \qquad v \in \mathcal{K}. \tag{16}$$

Then u satisfies

$$u(z)u(w)^* = \langle b(z), b(w)^*\rangle = 1 - \frac{1}{s_w(z)}, \tag{17}$$

and u is a contractive multiplier from $\mathcal{H}_s(\mathcal{K})$ to $\mathcal{H}_s$.

Proof. The point-wise adjoint of u satisfies

$$u(w)^*c = \overline{b(w)}c, \qquad w \in \mathcal{X}, c \in \mathbb{C}.$$

Hence $u(z)u(w)^* = \langle \overline{b(z)}, \overline{b(w)}\rangle = \langle b(z), b(w)^*\rangle$ by (15). The other equality follows from (14) and the identity

$$c = 1 - \frac{1}{1-c}, \qquad c \in \mathbb{C} \text{ with } |c| < 1.$$

To check that u is a contractive multiplier from $\mathcal{H}_s(\mathcal{K})$ to $\mathcal{H}_s$ we verify (13).

$$\begin{aligned} s_w(z) - u(z)u(w)^*s_w(z) &= [1 - u(z)u(w)^*]s_w(z) \\ &= [1 - (1 - \frac{1}{s_w(z)})]s_w(z) \\ &= 1 >> 0. \end{aligned}$$

□

Proposition 3.9. *A scalar-valued kernel s is a CNP kernel normalized at z_0 if and only if there exists an auxiliary Hilbert space $\mathcal{K}$ and a function $u : \mathcal{X} \to \mathcal{B}(\mathcal{K}, \mathbb{C})$, such that*

$$s_w(z) = \frac{1}{1 - u(z)u(w)^*} \tag{18}$$

with $u(z_0) = 0$.

Proof. Suppose first that s is given in the form (14), then Proposition 3.8 provides the construction of u.

Conversely suppose we have a Hilbert space $\mathcal{K}$ and a function $u : \mathcal{X} \to \mathcal{B}(\mathcal{K}, \mathbb{C})$, such that (18) holds. By the Riesz representation theorem, for each w in $\mathcal{X}$ there exists $\overline{b(w)}$ such that

$$u(z)v = \langle v, \overline{b(z)} \rangle \qquad \forall v \in \mathcal{K}.$$

Hence, by (15)

$$u(z)u(w)^* = \langle \overline{b(w)}, \overline{b(z)} \rangle = \langle b(z), b(w) \rangle.$$

□

Remark. From now on we will use the forms (14) and (18) interchangeably.

One important property of CNP kernels is that they can be re- normalized at any point and remain CNP kernels and, in addition, the multipliers of the spaces in question are preserved. In fact, this turns out to be a characterization of CNP kernels [10, Theorem 3.1], which will be used in the proof below. More information about about "scaling" kernels can be found in [3, Section 2.6].

Proposition 3.10. *Let s be a CNP kernel normalized at z_0. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Fix $z_1 \in \mathcal{X}$ and define the function $\tilde{s} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by*

$$\tilde{s}_w(z) = \frac{s(z_1, z_1)s(z, w)}{s(z, z_1)s(z_1, w)}. \tag{19}$$

Then $\tilde{s}$ is a CNP kernel normalized at z_1, and

$$\mathbf{Mult}(\mathcal{H}_{\tilde{s}}(\mathcal{L}_1), \mathcal{H}_{\tilde{s}}(\mathcal{L}_2)) = \mathbf{Mult}(\mathcal{H}_s(\mathcal{L}_1), \mathcal{H}_s(\mathcal{L}_2)),$$

with equality of norms.

Proof. Define a function $\delta : X \to \mathbb{C}$ by

$$\delta(z) = \frac{\sqrt{s(z_1, z_1)}}{s(z_1, z)}.$$

This gives $s_w(z)\overline{\delta(z)}\delta(w) = \tilde{s}_w(z)$, and so $\tilde{s}$ is a kernel by Proposition 2.18 and (7). To prove that $\tilde{s}$ is a CNP kernel normalized at z_1 we confirm that it satisfies (18). In general, every non-zero complex number c satisfies

$$c = \frac{1}{1 - (1 - \frac{1}{c})}.$$

Hence we only need to verify that $1 - \frac{1}{\tilde{s}} >> 0$, since $\tilde{s}$ is non-zero, but this is another characterization of CNP kernels, see [10, Theorem 3.1].

We now prove that multipliers of s and $\tilde{s}$ coincide. First note that both $\overline{\delta(z)}\delta(w) >> 0$ and $\frac{1}{\overline{\delta(z)}\delta(w)} >> 0$, by Proposition 2.18, since the latter equals

$$\frac{s(z, z_1)s(z_1, w)}{s(z_1, z_1)} = \frac{s(z, z_1)\overline{s(w, z_1)}}{s(z_1, z_1)}.$$

Suppose Φ is a multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_s(\mathcal{L}_2)$ with norm $\|M_\Phi\| \leq A$. From Proposition 2.31 we have the inequality

$$A^2 s_w(z) - \Phi(z)\Phi(w)^* s_w >> 0.$$

Furthermore since $\overline{\delta(z)}\delta(w) >> 0$, the above implies by (7) that

$$A^2 s_w(z)\overline{\delta(z)}\delta(w) - \Phi(z)\Phi(w)^* s_w(z)\overline{\delta(z)}\delta(w) >> 0,$$

this is equivalent to

$$A^2 \tilde{s}_w(z) - \Phi(z)\Phi(w)^* \tilde{s}_w(z) >> 0.$$

Hence Φ is a multiplier from $\mathcal{H}_{\tilde{s}}(\mathcal{L}_1)$ to $\mathcal{H}_{\tilde{s}}(\mathcal{L}_2)$ with $\|M_\Phi\| \leq A$. The other direction is proven similarly using the fact that $\frac{1}{\overline{\delta(z)}\delta(w)} >> 0$. □

Finally, we shall make use below of a special operator whose definition emerges from (14).

Definition 3.11. Let s be a normalized CNP kernel given in the form (14). For a Hilbert space $\mathcal{L}$ and a point $w \in \mathcal{X}$, let the operator $\tilde{E}^{\mathcal{H}}_{\overline{b(w)}} : \mathcal{L} \to \mathcal{K} \otimes \mathcal{L}$ be defined by

$$\tilde{E}^{\mathcal{L}}_{\overline{b(w)}} v = \overline{b(w)} \otimes v. \tag{20}$$

Sometimes, if the context is clear, we write

$$\tilde{E}^{\mathcal{L}}_{\overline{b(w)}} = \tilde{E}_{\overline{b(w)}} = \tilde{E}^{\mathcal{L}} = \tilde{E}.$$

Lemma 3.12. *Let w be a point in $\mathcal{X}$. Then the operator $\tilde{E}$ has the following properties:*

(a) $\tilde{E}^*_{\overline{b(z)}} \tilde{E}_{\overline{b(w)}} = \langle b(z), b(w)\rangle I_{\mathcal{K}}$.

(b) If $\mathcal{L}$ is a Hilbert space and $D : \mathcal{K} \otimes \mathcal{L} \to \mathcal{L}$ is a contraction, then the operator $I - D\tilde{E}^{\mathcal{L}}$ is invertible.

Proof. Firstly, the adjoint $\tilde{E}^*_{\overline{b(z)}}$ takes elements in $\mathcal{K} \otimes \mathcal{L}$, $u \otimes v$ to $\langle u, \overline{b(z)}\rangle v$. Hence $\tilde{E}^*_{\overline{b(z)}} \tilde{E}_{\overline{b(w)}} v = \langle \overline{b(w)}, \overline{b(z)}\rangle v$, which proves the equality by (15).

Secondly, note that $\|b(w)\| < 1$ by (14) and (3). Let v be a vector in $\mathcal{H}$. Then

$$\|\tilde{E}v\| = \|\overline{b(w)} \otimes v\| = \|\overline{b(w)}\|\|v\|$$

hence $\|\tilde{E}\| < \|b(w)\| < 1$. The second part is a well-known theorem in operator theory (see for example [11, Theorem 17.2]). □

3.3 Pick Interpolation on Arbitrary Subsets

This subsection concerns a special variant of the commutant lifting theorem, which we call Pick interpolation on arbitrary subsets. It is a special case of a general commutant-lifting theorem proved by Shimorin [4], who called it *the commutant lifting theorem of Nevanlinna-Pick type.*

In the following, we suppose that $\mathcal{Y}$ is a non-empty subset of $\mathcal{X}$ and s is a normalized CNP kernel on $\mathcal{X}$. Moreover, as before k will be a scalar-valued kernel on $\mathcal{X}$. For a Hilbert space $\mathcal{L}$, denote by $\mathcal{H}_k^{\mathcal{Y}}(\mathcal{L})$ the closed subspace of $\mathcal{H}_k(\mathcal{L})$ obtained from taking the closure of the linear span of vectors of form $k(\cdot, y)v, y \in \mathcal{Y}, v \in \mathcal{L}$. Furthermore let $k^{\mathcal{Y}} := k \restriction_{\mathcal{Y}\times\mathcal{Y}}$, which is a scalar-valued kernel on $\mathcal{Y}$.

Remark. The spaces $\mathcal{H}_k^{\mathcal{Y}}$ and $\mathcal{H}_{k^{\mathcal{Y}}}$ are different, as the first is space of functions on $\mathcal{X}$ and the second a space of functions on $\mathcal{Y}$.

Remark. If $z_0 \in \mathcal{Y}$ and k is normalized at z_0, then so is $k^{\mathcal{Y}}$.

Before stating the main result we need a definition followed by a proposition.

Definition 3.13. Let k be a scalar-valued kernel. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. We say that the linear operator $T : \mathcal{H}_k^{\mathcal{Y}}(\mathcal{L}_2) \to \mathcal{H}_k(\mathcal{L}_1)$ satisfies the *adjoint multiplier property* if for each $y \in \mathcal{Y}$ and $v \in \mathcal{L}_2$ there exists a $u \in \mathcal{L}_1$ such that

$$T[k(\cdot, y)v] = k(\cdot, y)u. \tag{21}$$

Proposition 3.14. *Let k be a scalar-valued kernel. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Suppose that the linear operator $T : \mathcal{H}_k^{\mathcal{Y}}(\mathcal{L}_2) \to \mathcal{H}_k(\mathcal{L}_1)$ is a contraction satisfying the adjoint multiplier property. Then the operator $\Psi^*(y)$ defined by $\Psi^*(y)v = u$, where y, u, and v are as in* (21)*, is a contractive linear operator from $\mathcal{L}_2$ to $\mathcal{L}_1$.*

Proof. The linearity of $\Psi^*(y)$ is inherited from T, this means that $\Psi^*(y)$ is well-defined. Furthermore since T is contractive, we have the inequality

$$\|k(\cdot, y)\Psi^*(y)v\| \leq \|k(\cdot, y)v\|,$$

which by (3) gives

$$\|\Psi^*(y)v\| \leq \|v\|.$$

□

Remark. We can now write equation (21) as

$$T[k(\cdot, y)v] = k(\cdot, y)\Psi(y)^*v, \qquad y \in \mathcal{Y}, v \in \mathcal{L}_2, \tag{22}$$

where $\Psi : \mathcal{Y} \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ is defined by $\Psi(\cdot) = (\Psi^*(\cdot))^*$ as in Proposition 3.14.

Our goal is to prove the following theorem which shows that operators satisfying (22) can be "lifted" to full multipliers of the space.

Theorem 3.15. *Let s be a normalized CNP kernel. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Suppose that the linear operator $T : \mathcal{H}_s^{\mathcal{Y}}(\mathcal{L}_2) \to \mathcal{H}_s(\mathcal{L}_1)$ is a contraction satisfying* (22)*. Then there exists a contractive multiplier $\tilde{\Psi}$ from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_s(\mathcal{L}_2)$ such that*

$$\tilde{\Psi}(y) = \Psi(y) \qquad \forall y \in \mathcal{Y}. \tag{23}$$

This directly gives $T = M_{\tilde{\Psi}}^* \restriction_{\mathcal{H}_s^{\mathcal{Y}}(\mathcal{L}_2)}$.

We will use the realization formula found in [3], stated below.

Theorem 3.16. *Let s be a normalized CNP kernel. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Then the $\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$-valued function Φ is a contractive multiplier from $\mathcal{H}_s(\mathcal{L}_1)$ to $\mathcal{H}_s(\mathcal{L}_2)$ if and only if there is an auxiliary Hilbert space $\mathcal{H}$ and an isometry $V : \mathcal{L}_2 \oplus \mathcal{K} \otimes \mathcal{H} \to \mathcal{L}_1 \oplus \mathcal{H}$ such that, writing V as*

$$V = \begin{array}{c} \\ \mathcal{L}_1 \\ \mathcal{H} \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{L}_2 & \mathcal{K} \otimes \mathcal{H} \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

one has

$$\Phi(w)^* = A + B\tilde{E}_{\overline{b(w)}}(I - D\tilde{E}_{\overline{b(w)}})^{-1}C.$$

The proof of Theorem 3.15 is based on some lemmas.

Lemma 3.17. *Let k be a scalar-valued kernel and $\mathcal{L}_2$ a Hilbert space. Let $y_1, \dots, y_N$ be some elements of $\mathcal{Y}$ and $v_1, \dots, v_N$ be some vectors in $\mathcal{L}_2$. Then*

$$\|\sum_{j=1}^N k^{\mathcal{Y}}(\cdot, y_j)v_j\|_{\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_2)} = \|\sum_{j=1}^N k(\cdot, y_j)v_j\|_{\mathcal{H}_k(\mathcal{L}_2)}. \tag{24}$$

Lemma 3.18. *Suppose s is a CNP kernel normalized at $z_0 \in \mathcal{Y}$ given in the form* (14)*. Then $s^{\mathcal{Y}}$ is a CNP kernel normalized at z_0.*

Proof. A kernel is a normalized complete Nevanlinna-Pick kernel if and only if it is given in the form (14). But in the case of $s^{\mathcal{Y}}$, b is restricted to $\mathcal{Y}$. ☐

Lemma 3.19. *Let k be a scalar-valued kernel. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Suppose $\Psi : \mathcal{Y} \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ is a given function. Then the linear operator T defined on the linear span of $\{k(\cdot, y)v : \ y \in \mathcal{Y}, v \in \mathcal{L}_2\}$ by the equation* (22) *extends to a contraction satisfying the adjoint multiplier property if and only if Ψ is a contractive multiplier from $\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_1)$ to $\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_2)$.*

Proof. Suppose T extends to a contraction. Define the operator $\tilde{T}$ on the linear span of $\{k^{\mathcal{Y}}(\cdot, y)v : \ y \in \mathcal{Y}, v \in \mathcal{L}_2\}$ by means of

$$\tilde{T}[k^{\mathcal{Y}}(\cdot, y)v] = k^{\mathcal{Y}}(\cdot, y)\Psi(y)^*v, \qquad y \in \mathcal{Y}, v \in \mathcal{L}_2.$$

We prove that $\tilde{T}$ extends to a contractive bounded linear operator from $\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_2)$ to $\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_1)$. Let $y_1, \dots, y_N$ be some lements of $\mathcal{Y}$ and $v_1, \dots, v_N$ some vectors in $\mathcal{L}_2$. Furthermore let $f = \sum_{i=j}^N k(\cdot, y_j)v_j$ and $\tilde{f} = \sum_{j=1}^N k^{\mathcal{Y}}(\cdot, y_i)v_i$. Then

$$\begin{aligned} \frac{\|\tilde{T}\tilde{f}\|_{\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_1)}}{\|\tilde{f}\|_{\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_2)}} &= \frac{\|\sum_{j=1}^N k^{\mathcal{Y}}(\cdot, y_i)\Psi(y_i)^*v_i\|_{\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_1)}}{\|\sum_{j=1}^N k^{\mathcal{Y}}(\cdot, y_i)v_i\|_{\mathcal{H}_{k^{\mathcal{Y}}}(\mathcal{L}_2)}} \\ &= \frac{\|\sum_{j=1}^N k(\cdot, y_i)\Psi(y_i)^*v_i\|_{\mathcal{H}_k(\mathcal{L}_1)}}{\|\sum_{j=1}^N k(\cdot, y_i)v_i\|_{\mathcal{H}_k(\mathcal{L}_2)}} \\ &= \frac{\|Tf\|_{\mathcal{H}_k(\mathcal{L}_1)}}{\|f\|_{\mathcal{H}_k(\mathcal{L}_2)}} \\ &\leq \|T\| \end{aligned}$$

by Lemma 3.17. Hence $\tilde{T}$ extends to a contraction, this implies that Ψ is a contractive multiplier by Proposition 2.30.

Conversely, the operator T has by definition the adjoint multiplier property. Moreover, $T = M_\Psi^*$ on a dense set of $\mathcal{H}_k^{\mathcal{Y}}(\mathcal{L}_2)$ by Proposition 2.30, and therefore extends to a contraction. □

Lastly, we verify that the adjoint multiplier property is invariant as to the point of normalization for a CNP kernel.

Proposition 3.20. *Let s be a CNP kernel normalized at z_0 and $\tilde{s}$ a re-normalization of s at a point $z_1 \in \mathcal{Y}$ as in Proposition 3.10. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. Suppose that the linear operator $T : \mathcal{H}_s^{\mathcal{Y}}(\mathcal{L}_2) \to \mathcal{H}_s(\mathcal{L}_1)$ is a contraction satisfying* (22)*. Then the operator R defined on the linear span of $\{\tilde{s}(\cdot, y)v : \ y \in \mathcal{Y}, v \in \mathcal{L}_2\}$ by means of*

$$R[\tilde{s}(\cdot, y)v] = \tilde{s}(\cdot, y)\Psi(y)^* v, \qquad y \in \mathcal{Y}, v \in \mathcal{L}_2$$

extends to a contractive linear operator from $\mathcal{H}_{\tilde{s}}^{\mathcal{Y}}(\mathcal{L}_2)$ to $\mathcal{H}_{\tilde{s}}(\mathcal{L}_1)$ and satisfies the adjoint multiplier property.

Proof. Ψ is a contractive multiplier from $\mathcal{H}_{s^{\mathcal{Y}}}(\mathcal{L}_1)$ to $\mathcal{H}_{s^{\mathcal{Y}}}(\mathcal{L}_2)$ by Lemma 3.19. Hence Lemma 3.18 and Proposition 3.10 implies that Ψ is also a contractive multiplier from $\mathcal{H}_{\tilde{s}^{\mathcal{Y}}}(\mathcal{L}_1)$ to $\mathcal{H}_{\tilde{s}^{\mathcal{Y}}}(\mathcal{L}_2)$. Therefore by Lemma 3.19, R extends to a contraction and satisfies the adjoint to a multiplier property. □

Now we are ready to prove the main result.

Proof of Theorem 3.15. We can assume without loss of generality that s is normalized at a point $z_0 \in \mathcal{Y}$. Since otherwise we can renormalize s at z_0 and look at the operator R as defined in Proposition 3.20.

Firstly, Ψ is a contractive multiplier from $\mathcal{H}_{s^{\mathcal{Y}}}(\mathcal{L}_1)$ to $\mathcal{H}_{s^{\mathcal{Y}}}(\mathcal{L}_2)$ by Lemma 3.19. Secondly, $s^{\mathcal{Y}}$ is a normalized CNP kernel by Lemma 3.18. Therefore by Theorem 3.16, Ψ is given by

$$\Psi(y)^* = A + B\tilde{E}_{\overline{b(y)}}(I - D\tilde{E}_{\overline{b(y)}})^{-1}C, \qquad y \in \mathcal{Y}.$$

Furthermore, the function

$$\tilde{\Psi}(w)^* := A + B\tilde{E}_{\overline{b(w)}}(I - D\tilde{E}_{\overline{b(w)}})^{-1}C, \qquad w \in X,$$

is well-defined by Lemma 3.12. Hence by Theorem 3.16, $\tilde{\Psi}$ is the multiplier that we desired. □

Having proved the above interpolation theorem, we now prove Leech's theorem. This particular version is due to Agler and McCarthy [3, Theorem 8.57]. An extension of this was proven by Shimorin [4, Corollary 2.3] and it is this approach which we follow.

Theorem 3.21 (Agler-McCarthy)**.** *Let s be a normalized CNP kernel. Let $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ be Hilbert spaces. Let $\Phi : \mathcal{Y} \to B(\mathcal{L}_1, \mathcal{L}_2)$ and $\Theta : \mathcal{Y} \to B(\mathcal{L}_3, \mathcal{L}_2)$ be given operator-valued functions. Then*

$$[\Phi(z)\Phi(w)^* - \Theta(z)\Theta(w)^*]s(z, w) >> 0 \tag{25}$$

if and only if there exists a contractive multiplier Ψ from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_s(\mathcal{L}_1)$, such that

$$\Phi(z)\Psi(z) = \Theta(z) \qquad \forall z \in \mathcal{Y}. \tag{26}$$

Proof. Firstly, suppose there exists a contractive multiplier Ψ from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_s(\mathcal{L}_1)$, such that (26) is satisfied. By (13)

$$[I_{\mathcal{L}_1} - \Psi(z)^*\Psi(w)^*]s_w >> 0,$$

this implies by Proposition 2.18 and (26) that

$$\begin{aligned}&[\Phi(z)\Phi(w)^* - \Theta(z)\Theta(w)^*]s_w(z)\\&= [\Phi(z)\Phi(w)^* - \Phi(z)\Psi(z)^*\Psi(w)^*\Phi(w)^*]s_w(z)\\&>> 0.\end{aligned}$$

Conversely, suppose that (25) is satisfied. Let $\mathcal{J}$ be the closure of the linear span of $\{\Phi(w)^*v : \ w \in \mathcal{Y}, v \in \mathcal{L}_2\}$, so $\mathcal{J}$ is a closed linear subspace of $\mathcal{L}_1$. Define a linear operator T on the linear span of $\{s_y^{\mathcal{Y}}v : \ y \in \mathcal{Y}, v \in \mathcal{L}_2\}$ by means of

$$T[s_w^Y\Phi(w)^*v] := s_w\Theta(w)^*v, \qquad w \in \mathcal{Y}, v \in \mathcal{L}_2.$$

Then T satisfies the adjoint multiplier property. Furthermore, the inequality (25) implies that T extends to a contraction from $\mathcal{H}_s^{\mathcal{Y}}(\mathcal{J})$ to $\mathcal{H}_s(\mathcal{L}_3)$. Indeed, for some elements $y_1, \ldots, y_N$ in $\mathcal{Y}$ and some vectors $v_1, \ldots, v_N$ in $\mathcal{L}_2$

$$\|T\sum_{j=1}^{N} s_{y_j}\Phi(y_j)^*v_j\| \leq \|\sum_{j=1}^{N} s_{y_j}\Phi(y_j)^*v_j\|.$$

Thus by Theorem 3.15, there exists a contractive multiplier Ψ from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_s(\mathcal{J})$ such that $T = M_\Psi^* \restriction_{\mathcal{H}_s^{\mathcal{Y}}(\mathcal{J})}$. This means that

$$s_w\Theta(w)^*v = s_w\Psi(w)^*\Phi(w)^*v, \qquad w \in \mathcal{Y}, v \in \mathcal{L}_2,$$

this implies, since s is non-zero that

$$\Theta(z) = \Phi(z)\Psi(z), \qquad z \in \mathcal{Y}.$$

Finally, note that Ψ is a contractive multiplier from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_s(\mathcal{L}_1)$ by Proposition 2.29. □

4 An Application

In this section we present a factorization theorem for reproducing kernel Hilbert spaces, where the kernel has a normalized CNP factor. This was first proved in [1] via a constructive method, but the authors also proposed a non-constructive approach, this is the one we will explore. The idea is to follow a method in [12] that used Theorem 3.21 to prove the factorization for Hilbert function spaces with normalized CNP kernel, but allow for vector-valued functions. Afterwards some techniques are used to extend this result to the case of kernels with a normalized CNP factor. Inspired by this, we will instead advance the usage of these techniques to prove a generalization of Theorem 3.21 that can then be used to more directly obtain the factorization.

4.1 Kernels with normalized CNP factor, Definition and Examples

Definition 4.1. We say that the scalar-valued kernel k on the set $\mathcal{X}$ is a kernel with *normalized CNP factor* if there exists two scalar-valued kernels g and s on $\mathcal{X}$, where s is a normalized CNP kernel, such that

$$k(z,w) = s(z,w)g(z,w), \qquad z,w \in \mathcal{X}. \tag{27}$$

Example 4.2. Of course one can take $g = 1$, and so the normalized CNP kernels $k = s$ is included among these spaces.

Example 4.3. The Bergman kernel is given by

$$k(z,w) = \frac{1}{(1 - z\overline{w})^2} = s^2(z,w),$$

where s denotes the Szegö kernel. So we have $k = sg$ with $g = s$.

Example 4.4. More generally, for any normalized CNP kernel s and $t \geq 1$ we have $k = s^t >> 0$. Note that if $s = \frac{1}{1-uu^*}$ and $0 < t < 1$, then

$$s^t = \sum_k a_k(t)(uu^*)^k,$$

with $a_k(t) > 0$, hence $s^t >> 0$. Then $s^t >> 0$ for all $t > 0$ by (7).

Example 4.5. The Hardy space $H^2(\mathbb{B}_d)$ is defined as the closure of analytic polynomials in $L^2(\sigma_d)$, where σ_d is the normalized Lebesgue measure on the unit sphere. Its reproducing kernel k is given by $k = s^d$, where s is the Drury-Arveson kernel.

In product domains, such as $\mathbb{D}^d, d \in \mathbb{N} \cup \{\infty\}$, the Hardy space $H^2(\mathbb{D}^d)$ is defined as the closure of analytic polynomials in $L^2(\sigma_1^2)$, where σ_1^d is the product of d copies of σ_1. In the case when $d = \infty$, we consider analytic polynomials in a finite number of variables. It is well known (see for example [7]), that $H^2(\mathbb{D}^\infty)$ can be identified with the space of Dirichlet series with square summable coefficients. It can be viewed as a reproducing kernel Hilbert space on the set Ω consisting of points in $\mathbb{D}^\infty$ whose coordinates form an ℓ^2-sequence. In all cases the reproducing kernel is given by

$$k_w(z) = \prod_{j=1}^{d} s^0_{w_j}(z_j), \qquad z = (z_j), w = (w_j),$$

where s^0 is the Szegö kernel. Clearly, each factor of this product is a normalized CNP factor of k

Example 4.6. *Weighted Bergman spaces.* Let μ be a finite positive measure on $\mathbb{B}_d, d \in \mathbb{N}$, such that for all $f \in \mathrm{Hol}(\mathbb{B}_d)$ and for any compact subset $K \subset \mathbb{B}_d$, there exists $c_K > 0$ such that

$$|f(z)|^2 \leq c_K \int_{\mathbb{B}_d} |f|^2 d\mu \qquad \forall z \in K.$$

The corresponding Bergman space $L^2_a(\mu) = L^2(\mu) \cap \mathrm{Hol}(\mathbb{B}_d)$ is a Hilbert space with reproducing kernel k^μ. In [1] the authors shows that $k^\mu / s >> 0$ for any analytic normalized CNP kernel in $\mathbb{B}_d$ such as the Drury-Arveson kernel.

4.2 Leech's theorem for kernels with normalized CNP factor

The techniques used to achieve the extension of Theorem 3.21 are contained in the lemmas below. Recall from section 2 that for a scalar-valued kernel g, $\mathcal{H}_g$ denotes the Hilbert function space that has g as its reproducing kernel. The idea is to decompose g and study it more closely.

Lemma 4.7. *Let k be a kernel with normalized CNP factor given in the form* (27)*. Let $\mathcal{D}_G = \mathcal{H}_g$ and define the function $G : \mathcal{X} \to \mathcal{B}(\mathcal{D}_G, \mathbb{C})$ by*

$$G(z)f = \langle f, g_z \rangle, \qquad f \in \mathcal{D}_G. \tag{28}$$

Then $G(z)G(w)^* = g_w(z)$.

Proof. The point-wise adjoint $G(w)^*$ takes a complex number c to $g_w c$. So then $G(z)G(w)^* = \langle g_w, g_z \rangle = g_w(z)$. □

The above lemma is sufficient to understand the scalar case, but we wish to investigate $\mathcal{H}_k(\mathcal{E})$ which has kernel $k_w I_\mathcal{E}$, and so a more delicate decomposition is required.

Lemma 4.8. *Let k be a kernel with normalized CNP factor given in the form* (27)*. Let $\mathcal{E}$ be a Hilbert space. Define the function $\tilde{G}_\mathcal{E} : X \to \mathcal{B}(\mathcal{D}_G \otimes \mathcal{E}, \mathcal{E})$ by*

$$\tilde{G}_\mathcal{E}(z)(v \otimes e) = (G(z)v)e, \qquad z \in \mathcal{X}, v \in \mathcal{D}_G, e \in \mathcal{E}. \tag{29}$$

Then $\tilde{G}_\mathcal{E}(z)\tilde{G}_\mathcal{E}(w)^* = I_\mathcal{E} g(z,w)$, $z, w \in \mathcal{X}$. Furthermore $\tilde{G}_\mathcal{E}$ is a contractive multiplier from $\mathcal{H}_s(\mathcal{D}_G \otimes \mathcal{E})$ to $\mathcal{H}_k(\mathcal{E})$ and $M^*_{\tilde{G}_\mathcal{E}}$ is an isometry.

Proof. The point-wise adjoint of $\tilde{G}_\mathcal{E}$ is given by

$$\tilde{G}_\mathcal{E}(w)^* v = (G(w)^* \otimes v), \qquad w \in \mathcal{X}, v \in \mathcal{E}.$$

So the operator T defined on the linear span of $\{k_w v : \ w \in \mathcal{X}, v \in \mathcal{E}\}$ by means of

$$T[k_w v] = s_w \tilde{G}_\mathcal{E}(w)^* v = s_w (G(w)^* \otimes v), \qquad w \in \mathcal{X}, v \in \mathcal{E}$$

extends to an isometric operator from $\mathcal{H}_k(\mathcal{E})$ to $\mathcal{H}_s(\mathcal{D}_G \otimes \mathcal{E})$ by (27). Hence $\tilde{G}_\mathcal{E}$ is a contractive multiplier from $\mathcal{H}_s(\mathcal{D}_G \otimes \mathcal{E})$ to $\mathcal{H}_k(\mathcal{E})$ by Proposition 2.31. □

An extension of Theorem 3.21 is now made.

Lemma 4.9. *Let k be a kernel with normalized CNP factor given in the form (27). Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{E}$ be Hilbert spaces. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$. Let $\Phi : \mathcal{Y} \to \mathcal{B}(\mathcal{L}_1 \oplus \mathcal{L}_2, \mathcal{E})$ with $\Phi = (\Phi_1, \Phi_2)$ and $\Theta : \mathcal{Y} \to \mathcal{B}(\mathcal{L}_3, \mathcal{E})$ be given operator-valued functions. Then*

$$[\Phi_1(z)\Phi_1(w)^* g_w(z) + \Phi_2(z)\Phi_2(w)^* - \Theta(z)\Theta(w)^*]s_w(z) >> 0 \tag{30}$$

if and only if there exists contractive multipliers Ψ_1 and Ψ_2 from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_k(\mathcal{L}_1)$ and $\mathcal{H}_s(\mathcal{L}_2)$, respectively, such that

$$\Phi_1(z)\Psi_1(z) + \Phi_2(z)\Psi_2(z) = \Theta(z) \qquad \forall z \in \mathcal{Y}. \tag{31}$$

Proof. Firstly, suppose there exists contractive multipliers Ψ_1 and Ψ_2 from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_k(\mathcal{L}_1)$ and $\mathcal{H}_s(\mathcal{L}_2)$, respectively, such that (31) is satisfied. Then

$$(\tilde{\Phi}_1(z), \Phi_2(z))(\Psi_1(z)\Psi_1(w)^* + \Psi_2(z)\Psi_2(w)^*) \begin{pmatrix} \tilde{\Phi}_1(w)^* \\ \Phi_2(w)^* \end{pmatrix} = \Theta(z)\Theta(w)^*. \tag{32}$$

Proposition 2.31 gives both

$$I_{\mathcal{L}_1} k_w(z) - \Psi_1(z)\Psi_2(w)^* s_w(z) >> 0$$

and

$$I_{\mathcal{L}_2} s_w(z) - \Psi_2(z)\Psi_2(w)^* s_w(z) >> 0.$$

Adding these together and multiplying by (Φ_1, Φ_2) as in Proposition 2.18, then using (32), we obtain (30).

Conversely, for convenience set $G_1 = \tilde{G}_{\mathcal{L}_1}$ as in Lemma 4.8. Moreover define $\Gamma(z) := (\Phi_1(z)G_1(z), \Phi_2(z))$, then by (30) and Lemma 4.8

$$[\Gamma(z)\Gamma(w)^* - \Theta(z)\Theta(w)^*]s_w(z) >> 0.$$

Hence there exists a contractive multiplier $\tilde{\Psi}$ from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_s((\mathcal{D}_G \otimes \mathcal{L}_1) \oplus \mathcal{L}_2)$, such that

$$\Gamma(z)\tilde{\Psi}(z) = \Theta(z) \qquad \forall z \in \mathcal{Y} \tag{33}$$

by Theorem 3.21. Furthermore by Lemma 2.33, $\tilde{\Psi}$ has factorization

$$\tilde{\Psi} = \begin{pmatrix} \tilde{\Psi}_1 \\ \Psi_2 \end{pmatrix},$$

where $\tilde{\Psi}_1$ and Ψ_2 are contractive multipliers from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_s(\mathcal{D}_G \otimes \mathcal{L}_1)$ and $\mathcal{H}_s(\mathcal{L}_2)$, respectively. Set $\Psi_1 = G_1\tilde{\Psi}_1$, then by Lemma 4.8 it is a contractive multiplier from $\mathcal{H}_s(\mathcal{L}_3)$ to $\mathcal{H}_k(\mathcal{L}_1)$. Finally, (33) gives

$$\begin{aligned}\Gamma(z)\tilde{\Psi}(z) &= (\Phi_1(z)G_1(z), \Phi_2(z)) \begin{pmatrix} \tilde{\Psi}_1(z) \\ \Psi_2(z) \end{pmatrix} \\ &= \Phi_1(z)G_1(z)\tilde{\Psi}_1(z) + \Phi_2(z))\Psi_2(z) \\ &= \Phi_1(z)\Psi_1(z) + \Phi_2(z)\Psi_2(z),\end{aligned}$$

which is what we set out to prove. □

Remark. This is a special case of [4, Corollary 2.3], specifically, setting $L = (g, 1)$ one obtains this lemma.

4.3 Proof of the factorization theorem

Before the proof of the factorization theorem we record a simple proposition.

Proposition 4.10. *Let F be a function in $\mathcal{H}_k(\mathcal{E})$ with $\|F\|_{\mathcal{H}_k(\mathcal{E})} \leq 1$. View $F(z)$ as a bounded linear operator from $\mathbb{C}$ to $\mathcal{E}$ defined by $F(z) : c \mapsto cF(z)$. Then*

$$I_{\mathcal{E}} k_w(z) - F(z)F(w)^* >> 0.$$

Proof. By Proposition 2.31 what is to be proved is equivalent to F being a multiplier from $\mathcal{H}_1(\mathbb{C})$ to $\mathcal{H}_k(\mathcal{E})$ with $\|M_F\| \leq 1$. But, $\mathcal{H}_1(\mathbb{C})$ is isomorphic to $\mathcal{H}_1$ by (9) which in turn is isomorphic to $\mathbb{C}$ by construction. Hence by Lemma 2.32, we only need to check that $F : c \mapsto cF(\cdot)$ defines a bounded linear map from $\mathbb{C}$ to $\mathcal{H}_k(\mathcal{E})$ with operator norm less than 1. However, this is true by assumption. □

We now turn to the factorization theorem.

Theorem 4.11. *For all $F : \mathcal{X} \to \mathcal{E}$ the following are equivalent:*

(a) $F \in \mathcal{H}_k(\mathcal{E})$ with $\|F\| \leq 1$,

(b) There is a $\psi \in Mult(\mathcal{H}_s)$ with $\psi(z_0) = 0$ and a $\Phi \in Mult(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ such that

$$\|\psi h\|^2_{\mathcal{H}_s} + \|\Phi h\|^2_{\mathcal{H}_k(\mathcal{E})} \leq \|h\|^2_{\mathcal{H}_s} \text{ for all } h \in \mathcal{H}_s,$$

and $F(z) = \frac{1}{1-\psi(z)}\Phi(z)$ for all $z \in \mathcal{X}$.

Proof of Theorem 4.11. Define the function $\Theta : X \to \mathcal{B}(\mathcal{E} \oplus \mathcal{K}, \mathcal{E})$ by

$$\Theta(z) = (\Theta_1(z), \Theta_2(z)) = (I_{\mathcal{E}}, F(z)u(z)).$$

For $w \in \mathcal{X}$ the point-wise adjoint of Θ is

$$\Theta(w)^* = \begin{pmatrix} \Theta_1(w)^* \\ \Theta_2(w)^* \end{pmatrix} = \begin{pmatrix} I_{\mathcal{E}} \\ u(w)^*F(w)^* \end{pmatrix},$$

where $F(w)$ is viewed as a bounded linear operator from $\mathbb{C}$ to $\mathcal{E}$. Thus

$$\Theta(z)\Theta(w)^* = \Theta_1(z)\Theta_1(w)^* + \Theta_2(z)\Theta_2(w)^* = I_{\mathcal{E}} + F(z)u(z)u(w)^*F(w)^*.$$

Therefore, from (17)

$$\begin{aligned}
&[\Theta_1(z)\Theta_1(w)^* g_w(z) + \Theta_2(z)\Theta_2(w)^* - F(z)F(w)^*]s_w(z) \\
&= k_w(z) + F(z)F(w)^*(1 - \frac{1}{s_w(z)})s_w(z) - F(z)F(w)^*s_w(z) \\
&= k_w(z) - F(z)F(w)^* \\
&>> 0
\end{aligned}$$

by Proposition 4.10. Hence Lemma 4.9 gives contractive multipliers Φ and ϕ from $\mathcal{H}_s$ to $\mathcal{H}_k(\mathcal{E})$ and $\mathcal{H}_s(\mathcal{K})$, respectively, such that

$$\Phi(z) + F(z)u(z)\phi(z) = F(z).$$

Set $\psi = u\phi$, this is a contractive multiplier in $\mathbf{Mult}(\mathcal{H}_s)$ by Proposition 3.8. The above equation yields

$$\begin{aligned} F(z) &= \Phi(z) + F(z)u(z)\phi(z) \\ \Leftrightarrow F(z)(1 - u(z)\phi(z)) &= \Phi(z) \\ \Leftrightarrow F(z) &= \frac{1}{1-\psi(z)}\Phi(z), \end{aligned}$$

which is well defined since ϕ is contractive and $\|u(z)\| < 1$ for all $z \in \mathcal{X}$. $\square$

References

[1] A. Aleman, M. Hartz, J. E. McCarthy, and S. Richter. "Factorizations induced by complete Nevanlinna–Pick factors." In: *Advances in Mathematics* 335 (2018), pp. 372–404. ISSN: 0001-8708.

[2] B. R. Leech. *Factorization of analytic functions, and operator inequalities.* URL: http://www.people.virginia.edu/~des5e/oldsotoa/S14,sotoa/Leech.pdf.

[3] J. Agler and J. E. McCarthy. *Pick Interpolation and Hilbert Function Spaces.* Vol. 44. Graduate Studies in Mathematics. 2002.

[4] S. Shimorin. "Commutant lifting and factorization of reproducing kernels." In: *Journal of Functional Analysis* 224.1 (2005), pp. 134–159. ISSN: 0022-1236.

[5] J. Conway. *A Course in Functional Analysis.* Vol. 96. Graduate Texts in Mathematics. 2007.

[6] "Bergman spaces" Compendium, Lund University.

[7] H. Hedenmalm, P. Lindqvist, and K. Seip. "A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$." In: *Duke Math. J.* 86.1 (1997), pp. 1–37.

[8] N. Wegge-Olsen. *K-Theory and C*-Algebras.* Oxford Science Publications. 1992.

[9] J. E. McCarthy and O. Moshe Shalit. "Spaces of Dirichlet series with the complete Pick property." In: *Israel Journal of Mathematics* 220 (2017), pp. 509–530.

[10] S. McCullough and T. T. Trent. "Invariant Subspaces and Nevanlinna–Pick Kernels." In: *Journal of Functional Analysis* 178.1 (2000), pp. 226–249. ISSN: 0022-1236.

[11] P. D. Lax. *Functional Analysis.* Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. John Wiley & Sons, Inc., 2002.

[12] A. Aleman, M. Hartz, J. McCarthy, and S. Richter. "The Smirnov class for spaces with the complete Pick property." In: *Journal of the London Mathematical Society* (Jan. 2017).